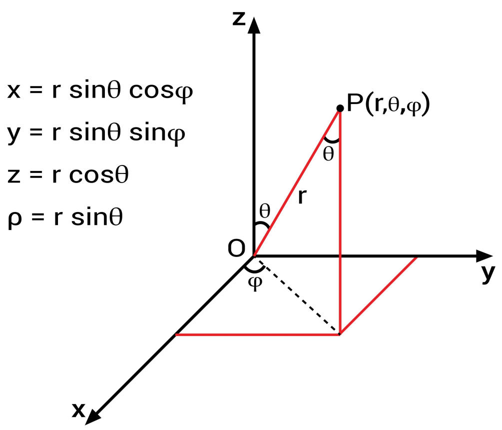
Gradient in Spherical Coordinates, Geometric Derivation

Conventions

NOTE

Note: There are different conventions for naming the angles (θ and ϕ). In physics, we use the standard ISO convention:

- r: The distance to the origin.
- θ (theta): The angle with the **z-axis** (polar angle, $0 \leq \theta \leq \pi$).
- ϕ (phi): The angle in the **xy-plane** (azimuth, $0 \leq \phi < 2\pi$).



Here is the derivation. To avoid writing out pages full of algebra using the chain rule (which becomes very complex in 3D), we use the **geometric approach** via "scale factors". This is much more intuitive and yields the same result.

The Intuition: Change per Distance

The gradient ∇f tells us how the function f changes per *physical distance* ($\mathrm{d}s$) traversed in a certain direction.

$$\text{Gradient component} \approx \frac{\text{change in} f}{\text{distance traveled} \mathrm{d}s}$$

In Cartesian coordinates this is simple, because $\mathrm{d}x$, $\mathrm{d}y$, and $\mathrm{d}z$ are already distances themselves. In spherical coordinates, $\mathrm{d}\theta$ and $\mathrm{d}\phi$ are *angles*, not distances. We therefore need to convert these angle changes into arc lengths.

Now use (is derived in the steps below)

$$\mathrm{d}s_r = \mathrm{d}r, \quad \mathrm{d}s_\theta = r\mathrm{d}\theta, \quad \mathrm{d}s_\phi = r\sin\theta\mathrm{d}\phi$$

then it follows that

$$
\begin{aligned}
(\nabla f)_r &= \frac{f(r+\mathrm{d}r,\theta,\phi)-f(r,\theta,\phi)}{\mathrm{d}s_r} = \frac{f(r+\mathrm{d}r,\theta,\phi)-f(r,\theta,\phi)}{\mathrm{d}r} = \frac{\partial f}{\partial r} \\
(\nabla f)_\theta &= \frac{f(r,\theta+\mathrm{d}\theta,\phi)-f(r,\theta,\phi)}{\mathrm{d}s_\theta} = \frac{f(r,\theta+\mathrm{d}\theta,\phi)-f(r,\theta,\phi)}{r\mathrm{d}\theta} = \frac{\partial f}{r\partial\theta} \\
(\nabla f)_\phi &= \frac{f(r,\theta,\phi+\mathrm{d}\phi)-f(r,\theta,\phi)}{\mathrm{d}s_\phi} = \frac{f(r,\theta,\phi+\mathrm{d}\phi)-f(r,\theta,\phi)}{r sin\theta\mathrm{d}\phi} = \frac{\partial f}{r sin\theta\partial\phi}
\end{aligned}
$$

So the result is

$$\nabla f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}\hat{\boldsymbol{\phi}}$$

Check by checking that the following is true

$$\mathrm{d}f = \nabla f \cdot \mathrm{d}\vec{s}$$

Verification

$$
\begin{aligned}
\nabla f \cdot \mathrm{d}\vec{s} &= (\nabla f)_r \mathrm{d}s_r + (\nabla f)_\theta \mathrm{d}s_\theta + (\nabla f)_\phi \mathrm{d}s_\phi \\
&= \frac{\partial f}{\partial r}\mathrm{d}r + \frac{1}{r}\frac{\partial f}{\partial\theta} r\mathrm{d}\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi} r\sin\theta\mathrm{d}\phi \\
&= \frac{\partial f}{\partial r}\mathrm{d}r + \frac{\partial f}{\partial\theta}\mathrm{d}\theta + \frac{\partial f}{\partial\phi}\mathrm{d}\phi \\
&= \mathrm{d}f
\end{aligned}
$$

Step 1: The radial direction (r)

If we vary a small step $\mathrm{d}r$ in the direction of $\hat{\mathbf{r}}$ (straight outwards):

- The distance traveled is simply the change in radius.
- $\mathrm{d}s_r = \mathrm{d}r$

Step 2: The polar angle direction (θ)

We vary the angle θ by a small amount $\mathrm{d}\theta$ (we move along a "meridian" downwards from the north pole):

- We move along a circle with radius r.
- The arc length is radius $\times$ angle.
- $\mathrm{d}s_\theta = r \cdot \mathrm{d}\theta$

Step 3: The azimuthal direction (ϕ)

We vary the angle ϕ by a small amount $\mathrm{d}\phi$ (we rotate around the z-axis):

- We move along a circle parallel to the xy-plane.
- **Note:** The radius of *this* circle is not r, but the distance from the point to the z-axis.
- If you look at the image, you can see that this radius is equal to $R_{xy} = r\sin\theta$.
- The arc length is therefore (radius in xy-plane) $\times$ angle.
- $\mathrm{d}s_\phi = (r\sin\theta) \cdot \mathrm{d}\phi$

Summary for your overview

Coordinate	**Unit Vector**	**Traversed Distance (ds)**	**Gradient Term (factor for the partial derivative)**
r	$\hat{\mathbf{r}}$	$\mathrm{d}r$	1
θ	$\hat{\boldsymbol{\theta}}$	$r\mathrm{d}\theta$	$\frac{1}{r}$
ϕ	$\hat{\boldsymbol{\phi}}$	$r\sin\theta\mathrm{d}\phi$	$\frac{1}{r\sin\theta}$